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Magnetograph Transformation in MFD

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The magnetograph transformation is introduced and employed to obtain solutions for plane, rotating, viscous, incompressible flows with orthogonal magnetic and velocity fields.

1. INTRODUCTION

A vast amount of research work has been done to analyze the motion of an electrically conducting fluid moving in a magnetic field since the early theoretical work of Alfvén. In recent years, the analysis of flow equations for the incompressible MHD flow of inviscid and viscous fluids having infinite electrical conductivity has been undertaken in various works.

In 1846 Hamilton coined the word "hodograph" for a velocity locus associated with a moving particle. If the components of velocity are $u_1(t)$, $u_2(t)$, and $u_3(t)$, the hodograph is the locus of a point whose position coordinates in an auxiliary space are $u_i(t)$, $i = 1, 2, 3$. This amounts to using the velocity components u_i as the independent variables in terms of which everything else, including the original position coordinates x_i, is to be expressed.

If the magnetic field vectors of an MHD fluid is laid off from a fixed point, the extremities of these vectors trace out a curve, called the magnetograph. Here we introduce a magnetograph transformation analogous to the hodograph one Chandra and Garg (1977) and obtain an equivalent linear system by interchanging the roles of dependent and independent variables.

Singh *et al.* (1986) determined the flow geometries when the velocity magnitude is constant along the individual streamlines. Gopal Krishna and Ramchandra Rao (1975) and Indrasena (1978) discussed the effect of rotation in a steady flow of an incompressible viscous fluid. Singh and Singh

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(1984) determined intrinsic relations and studied steady rotating hydromagnetic flows.

In the present work, using the magnetograph transformation, we obtain a linear partial differential equation of second order, a solution of which leads to the magnetic field of a flow. This approach is illustrated and two different solutions are obtained.

2. FLOW EQUATIONS

The steady flow in a rotating reference frame of a homogeneous, incompressible viscous fluid with infinite electrical conductivity is governed by the system of equations (Singh and Singh, 1984; Puri and Kulshesta, 1976)

$$\operatorname{div} \rho\bar{v} = 0 \tag{2.1}$$

$$\rho[(\bar{v}\cdot \operatorname{grad})\bar{v} + 2\bar{w}\times\bar{v} + \bar{w}\times(\bar{w}\times\bar{r})] = -\operatorname{grad} p + \eta\,\nabla^2\bar{v} + \mu \operatorname{curl}\bar{H}\times\bar{H} \tag{2.2}$$

$$\operatorname{curl}(\bar{v}\times\bar{H}) = 0 \tag{2.3}$$

$$\operatorname{div}\bar{H} = 0 \tag{2.4}$$

where $\bar{v}$ denotes the velocity vector, p the fluid pressure, $\bar{r}$ the radius vector, $\bar{w}$ the angular velocity vector, η the coefficient of viscosity, μ the constant magnetic permeability ρ the density, and $\bar{H}$ the magnetic field vector. In the case of plane flows with $\bar{H}$ in the plane of flow and orthogonal to the velocity vector $\bar{v}$, we have

$$uH_1 + vH_2 = 0 \tag{2.5}$$

where

$$\bar{V} = (u, v) \qquad \text{and} \qquad \bar{H} = (H_1, H_2)$$

From (2.3) we find that

$$uH_2 - vH_1 = K \tag{2.6}$$

where K is an arbitrary nonzero constant.

Equations (2.5) and (2.6) yield

$$u = \frac{KH_2}{H_1^2 + H_2^2}, \qquad v = \frac{-KH_1}{H_1^2 + H_2^2}$$

i.e.,

$$u = \frac{KH_2}{H^2}, \qquad v = \frac{-KH_1}{H^2} \tag{2.7}$$

where

$$H^2 = H_1^2 + H_2^2$$

Employing (2.7) in (2.1), we obtain

$$(H_2^2 - H_1^2)\frac{\partial H_1}{\partial y} + 2H_1H_2\left(\frac{\partial H_1}{\partial x} - \frac{\partial H_2}{\partial y}\right) + (H_2^2 - H_1^2)\frac{\partial H_2}{\partial x} = 0 \tag{2.8}$$

From equation (4), we have

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \tag{2.9}$$

Equations (2.8) and (2.9) form a system of two nonlinear partial differential equations in H_1 and H_2 by means of the magnetograph transformation, where H_1 and H_2 are regarded as independent variables with x and y as functions of H_1 and H_2. Equations (2.8) and (2.9) become

$$-(H_2^2 - H_1^2)\frac{\partial x}{\partial H_2} + 2H_1H_2\left(\frac{\partial y}{\partial H_2} - \frac{\partial x}{\partial H_1}\right) - (H_2^2 - H_1^2)\frac{\partial y}{\partial H_1} = 0 \tag{2.10}$$

$$\frac{\partial y}{\partial H_2} + \frac{\partial x}{\partial H_1} = 0 \tag{2.11}$$

provided that

$$\frac{\partial(H_1, H_2)}{\partial(x, y)} \neq 0$$

Equation (2.11) implies the existence of a function $\phi(H_1, H_2)$ such that

$$\frac{\partial \phi}{\partial H_1} = y, \qquad \frac{\partial \phi}{\partial H_2} = -x \tag{2.12}$$

Using (2.12) in (2.10), we get

$$(H_2^2 - H_1^2)\frac{\partial^2 \phi}{\partial H_2^2} + 4H_1H_2\frac{\partial^2 \phi}{\partial H_1\,\partial H_2} - (H_2^2 - H_1^2)\frac{\partial^2 \phi}{\partial H_1^2} = 0 \tag{2.13}$$

Introducing polar coordinates (H, θ) in the (H_1, H_2) plane we find that equation (2.13) transforms as

$$\frac{\partial^2 \phi}{\partial H^2} - \frac{1}{H^2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{H}\frac{\partial \phi}{\partial H} = 0 \tag{2.14}$$

Known solutions of (2.14) can be used to obtain some particular solutions for the plane orthogonal flows. Given a solution (H, θ) of (2.14), from (2.12) we have

$$x = -\frac{\partial \phi}{\partial H_2}, \qquad y = \frac{\partial \phi}{\partial H_1} \tag{2.15}$$

Knowing $x = x(H_1, H_2)$ and $y = y(H_1, H_2)$, we can express H_1 and H_2 as functions of x and y provided that $\partial(x, y)/\partial(H_1, H_2) \neq 0$. However, the velocity field thus obtained must satisfy the momentum equation (2.2), which may be written as

$$\begin{aligned} &\rho[(\text{curl } \bar{v} + 2\bar{w}) \times \bar{v}] + \text{grad}(P + \tfrac{1}{2}\rho \bar{V}^2) \\ &= \mu \text{ curl } \bar{H} \times \bar{H} - \eta \text{ curl curl } \bar{v} \end{aligned} \tag{2.16}$$

where $P = p - \frac{1}{2}\rho|\bar{w} \times \bar{v}|^2$ is the reduced pressure.

In the case of plane flows (2.16) gives us two scalar equations

$$\eta \frac{\partial \xi}{\partial y} - \rho \xi v - 2\rho w v + \mu j H_2 = -\frac{\partial B}{\partial x} \tag{2.17}$$

$$\eta \frac{\partial \xi}{\partial x} - \rho \xi u - 2\rho w u + \mu j H_1 = \frac{\partial B}{\partial y} \tag{2.18}$$

where $\bar{j}$ is the current density, $\bar{\xi}$ is the vorticity when the intensity of rotation is in the direction of the vorticity vector, and $B = P + \frac{1}{2}\rho v^2$ is the Bernoulli function.

3. SOLUTIONS

In this section we investigate two flow problems as applications of (2.14). Let a simple solution of (2.14) be given by

$$\phi = k_0 \theta = k_0 \tan^{-1}\left(\frac{H_2}{H_1}\right) \tag{3.1}$$

where k_0 is a positive constant.

From (2.15), we get

$$x = -\frac{\partial \phi}{\partial H_2} = -\frac{k_0 H_1}{H^2}, \qquad y = \frac{\partial \phi}{\partial H_1} = -\frac{k_0 H_2}{H^2} \tag{3.2}$$

which imply that

$$H_1 = -\frac{k_0}{r^2} x, \qquad H_2 = -\frac{k_0}{r^2} y \tag{3.3}$$

where $r^2 = x^2 + y^2$.

From (2.7), the velocity field $\bar{V}$ is given by

$$\bar{V}=(-ky/k_0,\ kx/k_0) \tag{3.4}$$

The vorticity $\bar{\xi}$ and current density $\bar{j}$ can be expressed as

$$\bar{\xi}=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{2k}{k_0} \tag{3.5}$$

$$\bar{j}=\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}=0 \tag{3.6}$$

From the integrability condition for B with the use of (2.17), (2.18), and (3.3)-(3.6), we obtain

$$x\frac{\partial w}{\partial y}-y\frac{\partial w}{\partial x}=0 \tag{3.7}$$

The most general solution of (3.7) is given by

$$w=y^2\,{}_1F_0\left[-1,\ldots,-\frac{x^2}{y^2}\right]=\sum_{n=0}^{\infty}\frac{(-1)_n(-1)^n x^{2n}y^{2-2n}}{\lfloor n} \tag{3.8}$$

where ${}_1F_0$ is the Gauss hypergeometric function (Slater, 1966). Taking $n=1$, we find that

$$w=c_1(x^2+y^2)=c_1r^2 \tag{3.9}$$

where c_1 is an arbitrary constant.

Using (3.3)-(3.6) and (3.9), we find that equations (2.17) and (2.18) reduce to

$$\begin{aligned}\frac{\partial B}{\partial x}&=\frac{2\rho K^2 x}{k_0^2}+\frac{2\rho c_1 K}{k_0}x(x^2+y^2)\\ \frac{\partial B}{\partial y}&=\frac{2\rho K^2}{k_0^2}y+\frac{2\rho c_1 K}{k_0}y(x^2+y^2)\end{aligned} \tag{3.10}$$

which implies that

$$B=\frac{\rho K^2}{k_0^2}r^2+\frac{\rho c_1 K}{k_0}\left(\frac{x^4+y^4}{2}+x^2y^2\right)+C_2 \tag{3.11}$$

where c_2 is an arbitrary constant, and the reduced pressure is given by

$$P=\frac{1}{2}\rho\frac{K^2}{k_0^2}r^2+\frac{\rho c_1 K}{k_0}\left(\frac{x^4+y^4}{2}+x^2y^2\right)+c_2 \tag{3.12}$$

Another simple solution of (2.14) is given by

$$\phi=\tfrac{1}{2}K_1(H_1^2+H_2^2)+K_2 \tag{3.13}$$

where K_1 and K_2 are arbitrary constants.

In this case we have

$$x = -\frac{\partial \phi}{\partial H_2} = -K_1 H_2, \qquad y = \frac{\partial \phi}{\partial H_1} = K_1 H_1 \tag{3.14}$$

and therefore the magnetic field is given by

$$H_1 = y/K_1, \qquad H_2 = -x/K_1 \tag{3.15}$$

These relations represents a circulatory flow. Further, we have

$$u = -\frac{xKK_1}{r^2}, \qquad v = -\frac{KK_1 y}{r^2}$$

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{3.16}$$

$$\bar{j} = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = -\frac{2}{K_1}$$

Using (3.16) in (2.17) and (2.18) and from the integrability condition for B, we have

$$y\frac{\partial w}{\partial y} + x\frac{\partial w}{\partial x} = 0 \tag{3.17}$$

The most general solution of (3.17) is given by

$$w = {}_2F_1\left[a;\, b;\, c;\, \frac{y}{x}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{\llcorner n (c)_n} \frac{y^n}{x^n}, \qquad x \neq 0 \tag{3.18}$$

where

$${}_2F_1[a;\, b;\, c;\, z] = 1 + \frac{ab}{c}\frac{z}{\llcorner 1} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{\llcorner 2} + \cdots$$

is the Gauss hypergeometric function (Slater, 1966), a, b, and c are constants $(c \neq 0)$, and

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \qquad \llcorner n = n(n-1)\cdots 1$$

Taking the most particular case when $n = 1$, we find that

$$w = c_3 y/x, \qquad x \neq 0 \tag{3.19}$$

where c_3 is an arbitrary constant.

Now (2.17) and (2.18) reduce respectively to

$$\frac{\partial B}{\partial x} = -2\rho KK_1 c_3 \frac{y^2}{x(x^2+y^2)} - \frac{2\mu}{K_1^2} x \tag{3.20}$$

$$\frac{\partial B}{\partial y} = 2\rho c_3 KK_1 \frac{y}{x^2+y^2} - \frac{2\mu}{k_1^2} y \tag{3.21}$$

which implies that

$$B = c_4 - \rho K K_1 c_3 \ln \frac{x^2}{r^2} - \frac{\mu}{K_1^2} r^2 \tag{3.22}$$

where c_4 is an arbitrary constant. Reduced pressure is given by

$$P = c_4 - c_3 \rho K K_1 \ln \frac{x^2}{r^2} - \frac{\mu}{K_1^2} r^2 - \frac{1}{2} \rho \frac{K K_1^2}{r^2} \tag{3.23}$$

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